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# Boost action on external field problems

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**Abstract.** We study the behaviour of chiral fermions in two-dimensional space-time with an external potential under the action of Lorentz boosts. Quantization of the system such that the incoming field is in the physical vacuum representation of the fermionic field algebra associated with the canonical anticommutation relations (CAR) algebra, makes time evolution and boosts unitarily implemented at all times. The models are shown to possess an instantaneous particle interpretation in exceptional cases only. In the cases they do, in general this property is destroyed by boosts. Thus boost invariance of instantaneous vacua, as has been conjectured recently, can be ruled out.

## 1. Introduction and conclusions

Particle number operators in general quantum field theory are derived from the basic objects of the theory, the quantum fields, in a rather indirect way. It is only via the associated asymptotic fields that such observables can be introduced. They represent the numbers of well separated particles which asymptotically constitute a scattering experiment. For non-free models no generally accepted particle number operators for intermediate times so far exist. This state of affairs has been considered disturbing and attempts have been made to identify such quantities and thereby establish an instantaneous particle interpretation (IPI) at least for linear external field problems.

For an otherwise free Dirac field in static external potentials the problem is solved by adopting for the algebra of canonical anticommutation relations (CAR) algebra the physical vacuum representation, which is associated with the static $c$-number dynamics to be quantized [1]. Further generalization to time dependent potentials has been proposed in [2–4]. The crucial object for the construction of an IPI is the instanteous vacuum: an instantaneous number operator exists if and only if (iff) the instantaneous vacuum does so [5].

Recently it has been argued (in a space-time dimension independent way) that instantaneous vacua are invariant under Lorentz boosts [6] and this has been advanced in support of their physical relevance. On the other hand, in the case of 4D spacetime this relevance was denied long ago by other authors [7], who reported instantaneous vacua to be changed by boosts.

Closer inspection of the two papers leaves it unresolved as to which one is right. The first [6] invokes Taylor expansions without having control over the terms to be dropped. In the latter [7] a proof is given that boosts are not implemented unitarily at the second quantized level, for non-zero regular external potentials. This fact however, neither rules out the existence of a boosted instantaneous vacuum nor does it imply its boost dependence, as has been observed in [7]. It is then stated in [7] without proof that instantaneous vacua are boost dependent.

The crucial criterion which needs to be checked is as follows [7]. The time $x^0$-vacuum of an external field problem with potential $A$ is invariant under a boost $\chi$ iff the one-particle space boost operator $L(x^0, \chi, A)$ intertwines between the positive spectral projections of the external field Hamiltonian $h(x^0, A)$ and the boosted one $h(x^0, \chi A)$:

$$\theta(h(x^0, \chi A))L(x^0, \chi, A) = L(x^0, \chi, A)\theta(h(x^0, A)).$$

This relation seems unlikely to be valid, since (even for $A = 0$) the boost operators do not intertwine in the naïve sense between the $c$-number Hamiltonians of the original and the boosted dynamics (cf equation (2.19)), i.e. $h(x^0, \chi A)L(x^0, \chi, A) \neq L(x^0, \chi, A)h(x^0, A)$. Nevertheless for zero external potential the projection-intertwining relation holds and in fact implies the well known boost invariance of the free field vacuum. One therefore needs to construct an explicit $A \neq 0$ counterexample to the projection-intertwining relation in order to confirm boost dependence of instantaneous vacua and refute their reported boost invariance. Here we will do so in a simple 2D setting.

We study the Lorentz transformation properties of the IPI for a class of very simple and explicitly solvable models. They describe chiral Dirac fermions in 2D space-time influenced by an external electromagnetic potential of compact support. Related models have been investigated in [8, 9]. Our findings are this: an IPI exists for very special external fields and then in general for few discrete instants of time only. This contrasts with the 4D case, where an IPI has been established for a large class of even irregular potentials [7]. Furthermore, whereas boosts are in general not unitarily implemented in 4D [7], they turn out to be implementable in our case. However, we find that they destroy the property of an (exceptional) model to possess an IPI. *A fortiori* boost invariance of instantaneous vacua can definitely be ruled out in 2D space-time, as a boosted vacuum in general does not even exist. Furthermore we make it explicit that existence of an IPI is gauge dependent.

## 2. The *c*-number model

Projecting the zero mass Dirac equation in 2D Minkowski space-time with a minimally coupled external electromagnetic potential onto its positive chirality part yields the following wave equation:

$$\mathrm{i}\frac{\partial}{\partial x^0}\Psi(x^0, x^1) = \left\{-\mathrm{i}\frac{\partial}{\partial x^1} - A(x^0, x^1)\right\}\Psi(x^0, x^1). \tag{2.1}$$

Here $A \in \mathscr{C}_0(\mathbb{R}^2 : \mathbb{R})$ (i.e. $A$ is continuous and of compact support) is assumed. The connection of $A$ with the Cartesian coordinate frame components of the electromagnetic potential is: $A = A_0 + A_1$. Equation (2.1) is interpreted quantum mechanically as an evolution equation in $\mathscr{H} := L^2(\mathbb{R} : \mathbb{C})$, the Hilbert space of Cauchy data to equation (2.1) with the usual scalar product:

$$(f, g) := \int_{\mathbb{R}} \mathrm{d}x^1 f(x^1)^* g(x^1).$$

The corresponding Hamiltonians we denote as follows:

$$h(x^0, A) := -\mathrm{i}\frac{\partial}{\partial x^1} - A(x^0, Q) \tag{2.2}$$

$$h_0 := -\mathrm{i}\frac{\partial}{\partial x^1}. \tag{2.3}$$

Here $Q$ is the multiplication operator $(Qf)(x) := xf(x)$.

The dynamics $\{u(x^0, A)/x^0 \in \mathbb{R}\}$ on $\mathscr{H}$, associated with (2.1) via the linear mapping $u(x^0, A) : \Psi(0; \cdot) \mapsto \Psi(x^0, \cdot)$, can be given explicitly:

$$u(x^0, A) = \mathrm{e}^{\mathrm{i}\gamma(x^0, Q)}\,\mathrm{e}^{-\mathrm{i}x^0 h_0} \tag{2.4}$$

$$\gamma(x^0, x^1) := \int_0^{x^0} \mathrm{d}\xi\, A(\xi, \xi + x^1 - x^0). \tag{2.5}$$

In order to prove equation (2.4) one simply checks that the given expression for $u(x^0, A)$ solves the defining evolution equation

$$\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}x^0} u(x^0, A) = h(x^0, A)u(x^0, A)$$

and obeys the initial condition $u(0, A) = Id_{\mathscr{H}}$.

The incoming Moller wave operator with respect to the free ($A = 0$) dynamics is defined by $W := s\text{-lim}\, u(t, A)^*\,\mathrm{e}^{-\mathrm{i}th_0}$ for $t \to -\infty$. From equations (2.4) and (2.5) we conclude that in terms of $w(x) := \int_{-\infty}^{0} \mathrm{d}\xi\, A(\xi, \xi + x)$ it reads

$$W = \mathrm{e}^{\mathrm{i}w(Q)}. \tag{2.6}$$

Obviously $W$ is unitary, a property we shall need later on. This unitarity can be traced back to the finite interaction time inherent in $A \in \mathscr{C}_0\,(\mathbb{R}^2 : \mathbb{R})$.

Let us now consider the action of the restricted Lorentz group $L_+^\uparrow \simeq (\mathbb{R}, +)$ of boosts on various objects of the $c$-number model. (For the sake of conceptual clarity we shall exclusively adopt the active point of view and never bring into the discussion a change of coordinates.) The primary group action is the boost representation of $(\mathbb{R}, +)$ on Minkowski space:

$$\chi x := \begin{pmatrix} (\chi x)^0 \\ (\chi x)^1 \end{pmatrix} := \begin{pmatrix} \cosh(\chi) & \sinh(\chi) \\ \sinh(\chi) & \cosh(\chi) \end{pmatrix} \begin{pmatrix} x^0 \\ x^1 \end{pmatrix}. \tag{2.7}$$

Associated with (2.7) is the chiral boost representation of $(\mathbb{R}, +)$ on the function space $\mathscr{C}^1(\mathbb{R}^2 : \mathbb{C})$ defined by

$$(\chi\Psi)(x) := \mathrm{e}^{\chi/2}\Psi(\chi^{-1}x). \tag{2.8}$$

Note that $\chi^{-1}x = (-\chi)x$. The factor $\mathrm{e}^{\chi/2}$ derives itself from the original spinor representation of the Lorentz group before projecting onto the positive chiral component and it will turn out to be decisive for obtaining unitary boost operators in $\mathscr{H}$.

For $A = 0$ only the representation (2.8) stabilizes the set of solutions to equation (2.1), while this is not the case for $A \neq 0$. The general case is this: iff $\Psi$ solves equation (2.1) then $\chi\Psi$ solves equation (2.1) with $A$ replaced by $\chi A$.

$$(\chi A)(x) := \mathrm{e}^{-\chi}A(\chi^{-1}x). \tag{2.9}$$

Therefore the Hamiltonian family generating the time evolution of $\chi\Psi$ is given by $\{h(x^0, \chi A)/x^0 \in \mathbb{R}\}$.

By means of formula (2.4) for the time evolution operators $u(x^0, A)$ for any $A \in \mathscr{C}_0(\mathbb{R}^2:\mathbb{R})$ we can derive an explicit formula for the boost operator $L(x^0, \chi, A)$ on $\mathscr{H}$, which is defined to map the $x^0$-Cauchy datum $\Psi(x^0, \cdot)$ of a solution to equation (2.1) onto the $x^0$-Cauchy datum $(\chi\Psi)(x^0, \cdot)$ of the boosted solution. To do so we express, by means of equation (2.4), an arbitrary solution to equation (2.1) in terms of its Cauchy datum at time zero:

$$\Psi(x^0, x^1) = e^{i\gamma(x^0, x^1)}\Psi(0, x^1 - x^0). \tag{2.10}$$

Now let $\chi$ operate on $\Psi$. By means of equation (2.10) we get

$$(\chi\Psi)(x^0, x^1) = e^{i(\chi\gamma)(x)}\, e^{\chi/2}\Psi(0, e^{\chi}(x^1 - x^0)). \tag{2.11}$$

Here $\gamma \in \mathscr{C}_0(\mathbb{R}^2:\mathbb{R})$ is given by equation (2.5) and $(\chi\gamma)(x) := \gamma(\chi^{-1}x)$. Observe

$$e^{\chi/2}\Psi(0, e^{\chi}(x^1 - x^0)) = \{e^{-ix^0 h_0}\Sigma(\chi)\Psi(0, \cdot)\}(x^1) \tag{2.12}$$

where $\Sigma(\chi)$ is the unitary operator on $\mathscr{H}$ defined by

$$(\Sigma(\chi)f)(x^1) := e^{\chi/2}f(e^{\chi}x^1). \tag{2.13}$$

$\Sigma(\chi)$ is known from non-relativistic quantum mechanics as the squeezing transformation [10] of the canonical pair $(Q, P := h_0)$:

$$\Sigma(\chi)^* Q\Sigma(\chi) = e^{-\chi}Q \qquad \Sigma(\chi)^* P\Sigma(\chi) = e^{\chi}P. \tag{2.14}$$

Equations (2.11) and (2.12) can be combined into

$$(\chi\Psi)(x) = \{e^{i(\chi\gamma)(x)}\, e^{-ix^0 h_0}\Sigma(\chi)\Psi(0, \cdot)\}(x^1). \tag{2.15}$$

Now we express $\Psi(0, \cdot)$ in (2.15) by means of equation (2.4) in terms of $\Psi(x^0, \cdot)$. Thus we obtain the boost operator explicitly as follows:

$$L(x^0, \chi, A) = e^{i(\chi\gamma)(x^0, Q)}\, e^{-ix^0 h_0}\Sigma(\chi)\, e^{ix^0 h_0}\, e^{-i\gamma(x^0, Q)}. \tag{2.16}$$

$L(x^0, \chi, A)$ is unitary due to the factor $e^{\chi/2}$ which appears in $\Sigma(\chi)$ and which can be traced back to the representation (2.8).

Obviously $L(x^0, \chi, A)$ depends on the external potential $A$ and $x^0$. For $A = 0$ the mapping $\chi \mapsto L(x^0, \chi, 0)$ constitutes a unitary representation of the boost group $(\mathbb{R}, +)$ on $\mathscr{H}$, which thus appears as a symmetry group of the model. For $A \neq 0$ it does not so: $L(x^0, \chi_2, A)L(x^0, \chi_1, A) \neq L(x^0, \chi_2 + \chi_1, A)$. On the energy momentum operators $P^0 := h_0 := P^1$ of the free model the boost operator $L(x^0, \chi, 0)$ implements, due to equation (2.14), the boost transformation of a right upper or left lower light-like vector:

$$L(x^0, \chi, 0)^* P^\mu L(x^0, \chi, 0) = e^{\chi}P^\mu. \tag{2.17}$$

From equation (2.17) the positive spectral projection intertwining relation

$$L(x^0, \chi, 0)^* \theta(P^0) L(x^0, \chi, 0) = \theta(P^0) \tag{2.18}$$

is immediate due to $e^{\chi} > 0$. For non-zero $A$ the boost operator $L(x^0, \chi, A)$ intertwines between $h(x^0, A)$ and $h(x^0, \chi A)$ in the following non-naïve way:

$$h(x^0, \chi A) = \left\{\left(i\frac{d}{dx^0} L(x^0, \chi, A)\right) + L(x^0, \chi, A)h(x^0, A)\right\} L(x^0, \chi, A)^*. \tag{2.19}$$

Equation (2.19) follows by differentiating the equation $u(x^0, \chi A)L(0, \chi, A) = L(x^0, \chi, A)u(x^0, A)$ with respect to $x^0$.

## 3. The second quantized model

The quantization of a linear spinorial wave equation like equation (2.1) may be done in an efficient and precise way by means of $C^*$-algebraic techniques [1, 7-9]. We shall do so. The primary concept is the CAR-algebra $\mathfrak{A}$ associated with the Hilbert space $\mathscr{H}$ of the $c$-number model and the canonical equal time anticommutation relations [11]. $\mathfrak{A}$ is generated by the image of $\mathscr{H}$ under an antilinear injection $a$ of $\mathscr{H}$ into $\mathfrak{A}$ and its unit element is denoted $e_{\mathfrak{A}}$. The anticommutation relations state that for any $f$, $g$ in $\mathscr{H}$ it holds that

$$\{a(f), a(g)^*\} = (f, g)e_{\mathfrak{A}} \qquad \{a(f), a(g)\} = 0.$$

Any orthogonal decomposition of $\mathscr{H} = \mathscr{H}^+ \oplus \mathscr{H}^-$ generates a quasifree, pure, gauge invariant state $\omega_P$ on $\mathfrak{A}$:

$$\omega_P(a(f_n)\dots a(f_1)a(g_1)^*\dots a(g_m)^*) := \delta_{nm}\det((f_i, Pg_j)). \tag{3.1}$$

Here $P$ is the orthogonal projection of $\mathscr{H}$ onto $\mathscr{H}^+$. The Gel'fand-Naimark-Segal (GNS) construction associates (up to isometrical equivalence) with $\omega_P$ a representation $\Pi_P: \mathfrak{A} \to \mathscr{L}(\mathscr{F}_P)$ of $\mathfrak{A}$ in the concrete $C^*$-algebra of linear bounded operators on a representation space $\mathscr{F}_P$. We denote $\Psi_P(f) := \Pi_P(a(f))$. Inspection of $\omega_{uPu^*}$, with $u$ being a unitary operator on $\mathscr{H}$, shows that the representation $a(f) \mapsto \Psi_P(u^*f)$ is isometrically equivalent with $\Pi_{uPu^*}$, i.e. there exists an isometry $\Gamma$: $\mathscr{F}_P \to \mathscr{F}_{uPu^*}$ with $\Gamma\Psi_P(u^*f) = \Psi_{uPu^*}(f)\Gamma$.

In order to construct a quantum field $\Psi$ which solves equation (2.1) in the distributional sense one may identify the restriction of $\Psi$ to an arbitrary initial time with $\Psi_P$ and let the time evolution (2.1) act on it in order to define a Heisenberg picture field. This amounts to defining for all $f$ in $\mathscr{H}$ (with initial time zero) [1]:

$$\Psi[x^0, f] := \Psi_P(u(x^0, A)^*f). \tag{3.2}$$

The proper choice of $P$ is drawn from the condition that $\Psi[x^0, \cdot]$ coincides with the free relativistic positive energy zero mass field at times before the external potential acts on it. Thus the Lehmann-Synzanzik-Zimmermann (LSZ) initial condition

$$\Psi_P(u(x^0, A)^*f) - \Psi_{P_0}(u(x^0, 0)^*f) = 0 \tag{3.3}$$

is assumed to hold for all $x^0 < -T < 0$ [7]. Here $T > 0$ is chosen to obey $\operatorname{supp}(A) \subset [-T, T] \times \mathbb{R}$ and $P_0 := \theta(h_0)$ is the positive spectral projection of the free $c$-number Hamiltonian. Equation (3.3) fixes $P := P_{\text{in}} := WP_0W^*$ with $W$ being the incoming wave operator from equation (2.6).

We shall now study the Heisenberg picture dynamics, which is encoded into the quantum field $\Psi[x^0, f] := \Psi_{P_{\text{in}}}(u(x^0, A)^*f)$. The prime question is whether an associated Schrödinger picture exists, i.e. whether there exists a unitary dynamics $\{\Gamma_{P\text{in}}(u(x^0, A))/x^0 \in \mathbb{R}\}$ on the representation space $\mathscr{F}_{P_{\text{in}}}$, which obeys for all $f$ in $\mathscr{H}$ and all $x^0$ in $\mathbb{R}$

$$\Gamma_{P_{\text{in}}}(u(x^0, A))^*\Psi_{P_{\text{in}}}(f)\Gamma_{P_{\text{in}}}(u(x^0, A)) = \Psi[x^0, f]. \tag{3.4}$$

To decide on the existence of $\Gamma_{P_{\text{in}}}(u(x^0, A))$ we observe that (for any orthogonal projection $P$ and any unitary $u$) a unitary $\Gamma_P(u)$ exists iff $\Pi_P$ and $\Pi_{uPu^*}$ are isometrically equivalent. Now two representations $\Pi_{P_1}$ and $\Pi_{P_2}$ are isometrically equivalent iff $P_1 - P_2 \in HS$, i.e. are Hilbert Schmidt operators [12]. Obviously $P_1 \sim P_2 \Leftrightarrow P_1 - P_2 \in HS$ defines an equivalence relation among the orthogonal projections of $\mathscr{H}$. (An equivalent condition to $P_1 - P_2 \in HS$ reads: $P_1(Id_{\mathscr{H}} - P_2) \in HS$ and $P_2(Id_{\mathscr{H}} - P_1) \in HS$ [13].) Thus $\Pi_{P_{\rm in}}$ and $\Pi_{u(x^0;A)P_{\rm in}u(x^0;A)^*}$ are isometrically equivalent iff $u(x^0, A)P_{\rm in}u(x^0, A)^* \sim P_{\rm in}$. Expressing $u(x^0, A)$ by (2.4) and $P_{\rm in} := WP_0W^*$ by (2.6) yields the equivalent condition $P_0 \sim e^{i\Delta(x^0,Q)}P_0 e^{-i\Delta(x^0,Q)}$ with $\Delta(x^0, Q) := \gamma(x^0, Q) - w(Q) + w(Q - x^0 Id_{\mathscr{H}})$. Whether *this latter condition is fulfilled can be decided for* $A \in \mathscr{C}_0^\infty(\mathbb{R}^2:\mathbb{R})$ *by means of the* following criterion [14]. Let $\alpha \in \mathscr{C}^\infty(\mathbb{R}:\mathbb{R})$ with $\alpha' \in \mathscr{C}_0^\infty(\mathbb{R}:\mathbb{R})$. Then it holds that $P_0 \sim e^{i\alpha(Q)}P_0 e^{-i\alpha(Q)} \Leftrightarrow \alpha(\infty) - \alpha(-\infty) \in 2\pi\mathbb{Z}$. It is immediate to check that, due to $A \in \mathscr{C}_0^\infty(\mathbb{R}^2:\mathbb{R})$, the functions $\gamma(x^0, \cdot)$ and $w(\cdot)$, and therefore $\Delta(x^0, \cdot)$ are also of the type $\mathscr{C}_0^\infty(\mathbb{R}:\mathbb{R})$. Thus the equivalence $e^{i\Delta(x^0,Q)}P_0 e^{-i\Delta(x^0,Q)} \sim P_0$ is seen to hold for all $x^0$. In this way we conclude that the Heisenberg picture time evolution $\Psi[0, \cdot] \mapsto \Psi[x^0, \cdot]$ is unitarily implemented at all times, thereby giving rise to an associated Schrödinger picture.

An (instantaneous) particle interpretation at time $x^0$ for a quantum field $\Psi$ with the dynamics $\{u(x^0, A)/x^0 \in \mathbb{R}\}$ is said to exist iff $\Psi[x^0, \cdot]$ is isometrically equivalent to $\Psi_{\theta(h(x^0,A))}(\cdot)$ [3, 4, 7]. An ample discussion of the physical reasoning which leads to this definition is given in [4]. The crucial fact is that the 'frozen' $c$-number dynamics $\{e^{-i(x^0-t)h(t,A)}/x^0 \in \mathbb{R}\}$, tangential to $\{u(x^0, A)/x^0 \in \mathbb{R}\}$ at time $t$, is unitarily implemented on $\Psi[t, \cdot]$ such that its generator is bounded from below and has a normalizable ground state vector, iff $\Psi[t, \cdot]$ is isometrically equivalent to $\Psi_{\theta(h(t,A))}(\cdot)$ [3, 15].

In order to see whether $\Psi$ has a particle interpretation at time $x^0$ we have to check $u(x^0, A)P_{\rm in}u(x^0, A)^* \sim \theta(h(x^0, A))$, or more explicitly $u(x^0, A)WP_0W^*u(x^0, A)^* \sim \theta(h(x^0, A))$. Since we already know from the unitary implementability of the time evolution that $u(x^0, A)WP_0W^*u(x^0, A)^* \sim P_0$ it suffices to check $P_0 \sim \theta(h(x^0, A))$. We *now make use of the unitary equivalence between* $h_0$ *and* $h(x^0, A)$ *which reads*

$$h(x^0, A) = e^{i\alpha(x^0,Q)}h_0 e^{-i\alpha(x^0,Q)} \tag{3.5}$$

$$\alpha(x^0, x^1) := \int_0^{x_1} d\xi\, A(x^0, \xi). \tag{3.6}$$

Thus $P_0 \sim \theta(h(x^0, A))$ amounts to $P_0 \sim e^{i\alpha(x^0,Q)}P_0 e^{-i\alpha(x^0,Q)}$. Applying the criterion of Hermaszewski and Streater [14], we obtain due to $A \in \mathscr{C}_0^\infty(\mathbb{R}^2:\mathbb{R})$: $P_0 \sim \theta(h(x^0, A)) \Leftrightarrow \int_{-\infty}^{+\infty} dx^1\, A(x^0, x^1) \in 2\pi\mathbb{Z}$. Therefore the field $\Psi$ has an IPI in the case of very special choices for $A$ and at discrete instants of time only. In particular the existence of an IPI is unstable under gauge transformations.

Iff $\Psi[x^0, \cdot]$ has a particle interpretation, the instantaneous vacuum $\Omega(x^0)$ with $\|\Omega(x^0)\| = 1$ exists. $\Omega(x^0)$ is the ground state to the second quantized dynamics, which is obtained by implementing $\{e^{-ih(x^0,A)t}/t \in \mathbb{R}\}$ on $\Psi[x^0, \cdot]$ and is, up to a phase, defined by decomposing $\Psi[x^0, \cdot]$ into instantaneous particle destruction and antiparticle creation parts $\Psi[x^0, f] = \Psi[x^0, \theta(h(x^0, A))f] + \Psi[x^0, \theta(-h(x^0, A))f]$. Then for all $f$ in $\mathscr{H}$ [3]:

$$\Psi[x^0, \theta(h(x^0, A))f]\Omega(x^0) = 0 \tag{3.7}$$

$$\Psi[x^0, \theta(-h(x^0, A))f]^*\Omega(x^0) = 0. \tag{3.8}$$

Note that due to equation (3.5) $Id_{\mathscr{H}} - \theta(h(x^0, A)) = \theta(-h(x^0, A))$.

We shall now study the action of a boost $\chi$ on the various objects of the second quantized model. The primary action is the boost action (2.8) on the quantum field $\Psi$. In the smeared form of the present formalism this reads

$$(\chi\Psi)[x^0, f] := \Psi[x^0, L(x^0, \chi, A)^* f]. \tag{3.9}$$

$\chi\Psi$ has the dynamics $\{u(x^0, \chi A)/x^0 \in \mathbb{R}\}$. Its incoming asymptotic field is, due to the compact support of $A$, again a free relativistic positive energy zero mass field.

Let us see whether $(\chi\Psi)[x^0, \cdot]$ and $\Psi[x^0, \cdot]$ are unitarily equivalent. To this end we have to check

$$L(x^0, \chi, A)u(x^0, A)\, WP_0W^*u(x^0, A)^*L(x^0, \chi, A)^* \sim u(x^0, A)\, WP_0W^*u(x^0, A)^*.$$

Since $u(x^0, A)\, WP_0W^*u(x^0, A)^* \sim P_0$, it suffices to check $L(x^0, \chi, A)P_0L(x^0, \chi, A)^* \sim P_0$. Due to the compact supports of $\gamma(x^0, \cdot)$ and $\chi\gamma(x^0, \cdot)$ this condition can, by means of equation (2.16), be further simplified to $L(x^0, \chi, 0)P_0L(x^0, \chi, 0)^* \sim P_0$. Equation (2.17) implies that a free boost $L(x^0, \chi, 0)$ stabilizes the positive and negative spectral parts of $h_0$ and therefore $L(x^0, \chi, 0)P_0L(x^0, \chi, 0)^* = P_0$ holds. Thus we have arrived at the result that for $A \in \mathscr{C}_0^\infty(\mathbb{R}^2 : \mathbb{R})$ boosts are unitarily implemented on the field $\Psi$ at all times. This stands in sharp contrast to the absence of implementability of boosts in the case of a massive Dirac field in 4D space-time with a non-vanishing regular external field [7].

Next we investigate the behaviour of an IPI under boosts. Let $A \in \mathscr{C}_0^\infty(\mathbb{R}^2 : \mathbb{R})$ obey for all $x^0$:

$$\int_{-\infty}^{+\infty} dx^1\, A(x^0, x^1) = 0. \tag{3.10}$$

Because $A$ is continuous, equation (3.10) is necessary and sufficient for $\Psi$ to have an IPI for all $x^0$. Since the boosted field $\chi\Psi$ has the dynamics $\{u(x^0, \chi A)/x^0 \in \mathbb{R}\}$, $\chi\Psi[x^0, \cdot]$ has an IPI at time $x^0$ if it is isometrically equivalent to $\Psi_{\theta(h(x^0, \chi A))}(\cdot)$. The boosted field at time $x^0$ is in the GNS-representation characterized by $UP_0U^*$ with $U := L(x^0, \chi, A)u(x^0, A)\, W$. Thus $\chi\Psi$ has an IPI at time $x^0$ iff

$$UP_0U^* \sim \theta(h(x^0, \chi A)). \tag{3.11}$$

Since we already know from the implementability of boosts that $UP_0U^* \sim P_0$, equation (3.11) is equivalent to $P_0 \sim \theta(h(x^0, \chi A))$, which holds iff

$$\int_{-\infty}^{+\infty} dx^1 (\chi A)(x^0, x^1) \in 2\pi\mathbb{Z}. \tag{3.12}$$

That equation (3.12) is in general not valid can be seen as follows. If (3.12) were to hold for all $x^0$ and $\chi$, the integral would have to be zero for all $x^0$ and $\chi$ by continuity. If we differentiate (3.12) with respect to $\chi$ at $x^0 = \chi = 0$ we obtain as a consequence $\int_{-\infty}^{+\infty} dx^1\, x^1(\partial/\partial x^0)A(0, x^1) = 0$. Take now for instance $A(x^0, x^1) := f(x^0)g(x^0 + x^1)$ with non-zero $f$, $g \in \mathscr{C}_0^\infty(\mathbb{R} : \mathbb{R})$ such that $f'(0) \neq 0$, $g(-x^1) = -g(x^1)$ and $g(x^1) \geqslant 0$ holds for $x^1 > 0$. Then equation (3.10) is valid for all $x^0$. Yet integrating $x^1(\partial/\partial x^0)A(0, x^1)$ yields, due to $x^1 g(x^1) \geqslant 0$,

$$\int_{-\infty}^{+\infty} dx^1\, x^1 \frac{\partial}{\partial x^0} A(0, x^1) = f'(0) \int_{-\infty}^{+\infty} dx^1\, x^1 g(x^1) \neq 0$$

which thus rules out equation (3.12). In this way we conclude that even for a potential which causes $\Psi$ to have an IPI for all times this property in general does not hold for the boosted field $\chi\Psi$.

Finally we comment on the exceptional choice for $A$ such that both $\Psi[x^0, \cdot]$ and $\chi\Psi[x^0, \cdot]$ have an IPI. Here one can study the relation between $\Omega(x^0)$, determined by equations (3.7) andd (3.8), and the instantaneous vacuum $\chi\Omega(x^0)$ of $\chi\Psi[x^0, \cdot]$. $\chi\Omega(x^0)$ obeys for all $f$ in $\mathscr{H}$:

$$(\chi\Psi)[x^0, \theta(h(x^0, \chi A))f]\chi\Omega(x^0) = 0 \tag{3.13}$$

$$(\chi\Psi)[x^0, \theta(-h(x^0, \chi A))f]^*\chi\Omega(x^0) = 0. \tag{3.14}$$

Equations (3.13) and (3.14) are by definition of $\chi\Psi$ equivalent to the following ones, which hold for all $f$ in $\mathscr{H}$:

$$\Psi[x^0, L(x^0, \chi, A)^*\theta(h(x^0, \chi A))f]\chi\Omega(x^0) = 0 \tag{3.15}$$

$$\Psi[x^0, L(x^0, \chi, A)^*\theta(-h(x^0, \chi A))f]^*\chi\Omega(x^0) = 0. \tag{3.16}$$

The boosted vacuum $\chi\Omega(x^0)$ coincides with $\Omega(x^0)$ (up to a phase) iff the positive spectral projections involved in (3.7) and (3.15) coincide. Thus the condition for boost invariance of $\Omega(x^0)$ reads

$$L(x^0, \chi, A)^*\theta(h(x^0, \chi A))L(x^0, \chi, A) = \theta(h(x^0, A)). \tag{3.17}$$

Since $\theta(h(x^0, A)) \sim P_0$ and $L(x^0, \chi, A)P_0L(x^0, \chi, A)^* \sim P_0$ equation (3.17) has a chance of being fulfilled only in the case $\theta(h(x^0, \chi A)) \sim P_0$. This latter equivalence, however, has been ruled out as being generally valid.

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